JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **25**, No. 2, May 2012

POWER INEQUALITY ON THE SIMPLEX

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Abstract. The power inequality $\prod_{k=1}^{N} x_k^{x_k} \geq \prod_{k=1}^{N} p_k^{x_k}$ holds for the points $(x_1, ..., x_N)$,$(p_1, ..., p_N)$ of the simplex. We show this using the analytic method combining Frostman's density theorem with the strong law of large numbers.

1. Introduction

The power inequality on the simplex is sometimes interesting by itself. Recently, we([1]) studied the Hausdorff dimensions of the local dimension sets and the distribution sets of a self-similar Cantor set using the strong law of large numbers together with the Frostman's density theorem. This can be also generalized to the dimensions of the the local dimension sets and the distribution sets of a self-similar set([3]). The self-similar set is a generalized form of the self-similar Cantor set in the sense that the self-similar set is generated by N similarities with an integer $N \geq 2$ while the self-similar Cantor set is generated by 2 similarities. The main idea to get these results is that the distribution set gives the exact local dimension for the self-similar measure on the self-similar set. The connection between the distribution set and the local dimension set of the self-similar set is that the frequencies of digits of the N-ary code which corresponds to a point of the self-similar set can be transformed into the log-density of the probability of the cylinder of the code by its diameter. We use the strong law of large numbers for the lower bound for the Hausdorff dimension of the distribution set which is contained in the local dimension set. In this paper, we give some constraint which generates a hyperplane of the simplex. This constraint plays a role

Received February 14, 2012; Accepted April 17, 2012.
2010 Mathematics Subject Classification: Primary 28A78.
Key words and phrases: simplex, strong law of large numbers, density theorem, Hausdorff dimension.
This work was supported by the research grant of the Busan University of Foreign Studies in 2012.

for the distribution set to be a subset of the local dimension set of the self-similar measure or the probability on the self-similar set. The monotonicity of the Hausdorff dimension gives our essential inequality. That is, the points of the hyperplane by a constraint of the simplex satisfy our essential inequality, which gives the power inequality. Our result is a byproduct of the study([3]) of the relation between the local dimension set and distribution set of the self-similar set.

2. Preliminaries

Let $\mathbb{N}$ and $\mathbb{R}$ be the set of positive integers and the set of real numbers respectively. An attractor K in the d-dimensional Euclidean space $\mathbb{R}^d$ of the iterated function system(IFS) $(f_1, ..., f_N)$ of contractions where $N \geq 2$ makes each point $q \in K$ have an infinite sequence $\omega = (m_1, i_2, ...) \in \Sigma = \{1, ..., N\}^{\mathbb{N}}$ where

$$\{q\} = \bigcap_{n=1}^{\infty} K_{\omega|n}$$

for $K_{\omega|n} = K_{m_1,...,m_n} = f_{m_1} \circ \cdots \circ f_{m_n}(K)$([2, 4, 5]). In such case, we sometimes write $\pi(\omega)$ for such q using the natural projection $\pi : \Sigma \to K$ and write $c_n(q)$ for such $K_{\omega|n}$. We call such $c_n(q)$ n-th cylinder of K and $|c_n(q)|$ denotes the diameter of $c_n(q)$ and we also call ω the code of q.

Consider a probability vector $\mathbf{p} = (p_1, ..., p_N)$([5]). Each infinite sequence $\omega = (m_1, m_2, ...)$ has the unique subset $A(y_n(\omega))$ of its accumulation points in the simplex of probability vectors in $\mathbb{R}^N$ of the vector-valued sequence $\{y_n(\omega)\} = \{(p_1, ..., p_N)_n\}$ of the probability vectors where p_k for $1 \leq k \leq N$ in the probability vector $(p_1, ..., p_N)_n$ for each $n \in \mathbb{N}$ is defined by

$$p_k = \frac{|\{1 \leq l \leq n : m_l = k\}|}{n}.$$

Sometimes we write $n_k(\omega|n)$ for such p_k. $\dim(E)$ denotes the Hausdorff dimension of E([4]). For the self-similar measure $\gamma_{\mathbf{p}}$ on K associated with $\mathbf{p} = (p_1, ..., p_N) \in (0,1)^N$ satisfying $\sum_{i=1}^N p_i = 1$, we write $E_\alpha^{(\mathbf{p})^*}$ for the set of points at which the local dimension of $\gamma_{\mathbf{p}}$ on K is exactly α, so that

$$E_\alpha^{(\mathbf{p})^*} = \{q \in K : \lim_{r \to 0} \frac{\log \gamma_{\mathbf{p}}(B_r(q))}{\log r} = \alpha\}. \tag{1}$$

We call $\{E_\alpha^{(\mathbf{p})^*}(\neq \phi) : \alpha \in \mathbb{R}\}$ the spectral class generated by the local dimensions of a self-similar measure $\gamma_{\mathbf{p}}$. We call α an associated local dimension of $\gamma_{\mathbf{p}}$. We define the cylinder local dimension set

$$E_\alpha^{(\mathbf{p})} = \{q \in K : \lim_{n\to\infty} \frac{\log \gamma_{\mathbf{p}}(c_n(q))}{\log |c_n(q)|} = \alpha\}. \tag{2}$$

It is well-known([1, 2, 5]) that if the IFS $(f_1, ..., f_N)$ satisfies the strong separation condition(SSC) then

$$E_\alpha^{(\mathbf{p})^*} = E_\alpha^{(\mathbf{p})}.$$

In this paper, we assume that the IFS satisfies the strong separation condition (SSC)([2, 4, 5]). We mainly discuss the cylinder local dimension sets $E_\alpha^{(\mathbf{p})}$ instead of $E_\alpha^{(\mathbf{p})^*}$ for studying the attractor K of the IFS satisfying SSC since the cylinder local dimension sets are quite closely related to the distribution sets. In this paper, we assume that $0 \log 0 = 0$ for convenience.

3. Main results

We have the following essential inequality for the points of the hyperplane(by a constraint) of the simplex. We note that there is no measure theoretical assumption in the Theorem. However we use the multifractal or measure theoretical technique to prove this theorem. We arrange some Lemma and Propositions after this Theorem to emphasize this.

THEOREM 3.1. *Let* $(p_1, ..., p_N), (a_1, ..., a_N) \in [0, 1]^N$ *and* $\sum_{k=1}^N p_k = 1$.

For $(x_1, ..., x_N) \in [0, 1]^N$ *with* $\sum_{k=1}^N x_k = 1$ *satisfying the equation*

$$\frac{\sum_{k=1}^N x_k \log p_k}{\sum_{k=1}^N x_k \log a_k} = \frac{\sum_{k=1}^N p_k \log p_k}{\sum_{k=1}^N p_k \log a_k}, \tag{3}$$

we have

$$\frac{\sum_{k=1}^N x_k \log x_k}{\sum_{k=1}^N x_k \log a_k} \leq \frac{\sum_{k=1}^N p_k \log p_k}{\sum_{k=1}^N p_k \log a_k}. \tag{4}$$

Proof. It suffices to show that it holds for $(p_1, ..., p_N), (a_1, ..., a_N)$, $(x_1, ..., x_N) \in (0, 1)^N$ with $\sum_{k=1}^N p_k = \sum_{k=1}^N x_k = 1$. Let

$$\mathbf{p} = (p_1, ..., p_N),$$

$$\mathbf{x} = (x_1, ..., x_N).$$

Define

$$g(\mathbf{x},\mathbf{p}) = \frac{\sum_{k=1}^{N} x_k \log p_k}{\sum_{k=1}^{N} x_k \log a_k}. \tag{5}$$

We only need to show that if $g(\mathbf{x},\mathbf{p}) = g(\mathbf{p},\mathbf{p})$, then $g(\mathbf{x},\mathbf{x}) \le g(\mathbf{p},\mathbf{p})$. It follows from our last Proposition. □

From now on, we assume that the similarity ratios of the similarities $(f_1,...,f_N)$ are $a_1,...,a_N \in (0,1)$ and K is the self-similar set for the IFS $(f_1,...,f_N)$ and $\gamma_{\mathbf{p}}$ on K is the self-similar measure associated with $\mathbf{p} = (p_1,...,p_N) \in (0,1)^N$ satisfying $\sum_{k=1}^{N} p_k = 1$.

Lemma 3.2. *If $E(\subset K)$ satisfies $\gamma_{\mathbf{p}}(E) > 0$ and $\lim_{n\to\infty} \frac{\log \gamma_{\mathbf{p}}(c_n(q))}{\log |c_n(q)|} = \alpha$ for all $q \in E$, then $\dim(E) = \alpha$.*

Proof. It follows from the Frostman's density theorem([4]). □

Proposition 3.3. *Let the distribution set*

$$F(\mathbf{x}) = \{\omega : \lim_{n\to\infty} n_k(\omega|n) = x_k, 1 \le k \le N\}. \tag{6}$$

Then

$$\dim(\pi(F(\mathbf{x}))) = g(\mathbf{x},\mathbf{x}).$$

Further

$$\dim(E^{(\mathbf{p})}_{g(\mathbf{p},\mathbf{p})}) = g(\mathbf{p},\mathbf{p}).$$

Proof. $\gamma_{\mathbf{x}}(\pi(F(\mathbf{x}))) = 1$ follows from the strong law of large numbers and

$$\lim_{n\to\infty} \frac{\log \gamma_{\mathbf{x}}(c_n(q))}{\log |c_n(q)|} = g(\mathbf{x},\mathbf{x})$$

for all $q \in \pi(F(\mathbf{x}))$. For, since $q = \pi(\omega)$ where $\omega \in F(\mathbf{x})$,

$$\frac{\log \gamma_{\mathbf{x}}(c_n(q))}{\log |c_n(q)|} = \frac{\sum_{k=1}^{N} n_k(\omega|n) \log x_k}{\sum_{k=1}^{N} n_k(\omega|n) \log a_k},$$

so

$$\lim_{n\to\infty} \frac{\log \gamma_{\mathbf{x}}(c_n(q))}{\log |c_n(q)|} = \frac{\lim_{n\to\infty} \sum_{k=1}^{N} n_k(\omega|n) \log x_k}{\lim_{n\to\infty} \sum_{k=1}^{N} n_k(\omega|n) \log a_k} = g(\mathbf{x},\mathbf{x}),$$

from (5). Similarly

$$\pi(F(\mathbf{p})) \subset E^{(\mathbf{p})}_{g(\mathbf{p},\mathbf{p})},$$

which gives $\gamma_{\mathbf{p}}(E^{(\mathbf{p})}_{g(\mathbf{p},\mathbf{p})}) = 1$. By the definition of (2),

$$\lim_{n\to\infty} \frac{\log \gamma_{\mathbf{p}}(c_n(q))}{\log |c_n(q)|} = g(\mathbf{p},\mathbf{p})$$

for all $q \in E^{(\mathbf{p})}_{g(\mathbf{p},\mathbf{p})}$. It follows from the above Lemma. □

Remark 3.4. We note that in (6)

$$F(\mathbf{x}) = \{\omega : A(y_n(\omega)) = \{\mathbf{x}\}\}.$$

Proposition 3.5. *If (3) is satisfied, then*

$$\pi(F(\mathbf{x})) \subset E^{(\mathbf{p})}_{g(\mathbf{p},\mathbf{p})}.$$

Further

$$g(\mathbf{x}, \mathbf{x}) \leq g(\mathbf{p}, \mathbf{p}).$$

Proof. $\pi(F(\mathbf{x})) \subset E^{(\mathbf{p})}_{g(\mathbf{p},\mathbf{p})}$ follows from the assumption $g(\mathbf{x}, \mathbf{p}) = g(\mathbf{p}, \mathbf{p})$. It follows from the above Proposition. □

Remark 3.6. $\mathbf{x}$ satisfying the condition (3) form a hyperplane of the simplex except for $\mathbf{p} = \mathbf{p}_s \equiv (a_1^s, ..., a_N^s)$ where $\sum_{k=1}^N a_k^s = 1$. That is, by the definition of (5), $\{\mathbf{x} : g(\mathbf{x}, \mathbf{p}) = g(\mathbf{p}, \mathbf{p})\}$ is a hyperplane of the simplex except for the singular point $\mathbf{p} = \mathbf{p}_s$. For $\mathbf{p} = \mathbf{p}_s$, the condition (3) is not a constraint but a tautology. However the inequality (4) still holds for whole points $\mathbf{x}$ of the simplex. We also note that

$$\max\{g(\mathbf{x}, \mathbf{x}) : g(\mathbf{x}, \mathbf{p}) = g(\mathbf{p}, \mathbf{p})\} = g(\mathbf{p}, \mathbf{p}).$$

Our essential inequality (4) gives the following power inequality which is our main Theorem.

Theorem 3.7. *For the points $(x_1, ..., x_N)$, $(p_1, ..., p_N)$ of the $(N-1)$-simplex, that is, for $(x_1, ..., x_N), (p_1, ..., p_N) \in [0, 1]^N$ with $\sum_{k=1}^N x_k = \sum_{k=1}^N p_k = 1$, we have*

$$\prod_{k=1}^N x_k^{x_k} \geq \prod_{k=1}^N p_k^{x_k}. \tag{7}$$

Proof. For $(p_1, ..., p_N), (x_1, ..., x_N) \in [0, 1]^N$ with $\sum_{k=1}^N p_k = \sum_{k=1}^N x_k = 1$, it is not difficult to show that there is $(a_1, ..., a_N) \in (0, 1)^N$ such that $g(\mathbf{x}, \mathbf{p}) = g(\mathbf{p}, \mathbf{p})$, where $g(\mathbf{x}, \mathbf{p})$ is defined by (5) for

$$\mathbf{p} = (p_1, ..., p_N),$$

and

$$\mathbf{x} = (x_1, ..., x_N).$$

From (3) and (4), we have

$$\frac{\sum_{k=1}^N x_k \log x_k}{\sum_{k=1}^N x_k \log a_k} \leq \frac{\sum_{k=1}^N p_k \log p_k}{\sum_{k=1}^N p_k \log a_k} = \frac{\sum_{k=1}^N x_k \log p_k}{\sum_{k=1}^N x_k \log a_k}.$$

This gives the inequality

$$\sum_{k=1}^{N} x_k \log x_k \geq \sum_{k=1}^{N} x_k \log p_k \tag{8}$$

since $\sum_{k=1}^{N} x_k \log a_k < 0$. (7) follows from (8). $\square$

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